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Decomposition of Linear Systems in Dual Flow Problems

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1. Statement of the problem.

Consider a final oriented generalized network $G = (I, U)$, $|U| \gg |I|$, with a set of nodes I and a set of arcs U, without multiple arcs and loops. Let $I^* \subseteq I$ be the subset of nodes with variable intensities $\pm x_i$, $I^* \neq \oslash$, $sign(i) = 1$, if $i \in I^*_+$, $sign(i) = -1$, if $i \in I^*_-$, I^*_+, $I^*_- \subseteq I^*$, $I^*_+ \cap I^*_- = \oslash$. $I \setminus I^*$ is the subset of nodes with constant intensities a_i, $i \in I \setminus I^*$. Let us introduce the characteristic c_i for nodes $i \in I^*$, which means the expense of increasing the manufacturing of product by one unit, if $i \in I^*_+$, and the expense of storing a unit of product, if $i \in I^*_-$. Other characteristics are traditional: d_{ij} – capacity of arc (i, j); x_{ij} – a flow along the same arc; c_{ij} – cost of a unit transportation of flow along arc (i, j), μ_{ij} – a flow transformation coefficient for arc (i, j), $\mu_{ij} \in]0, 1]$, $(i, j) \in U$, $I^+_i(U) = \{j : (i, j) \in U\}$, $I^-_i(U) = \{j : (j, i) \in U\}$, b_{*i}, b^*_i – the lower and upper bounds of the intensity x_i, $i \in I$. Consider the following extreme problem on the generalized network G:

$$(1) \qquad \varphi(x) = \sum_{(i,j)\in U} c_{ij}x_{ij} + \sum_{i\in I^*} c_i x_i \longrightarrow \min,$$

$$(2) \qquad \sum_{j\in I^+_i(U)} x_{ij} - \sum_{j\in I^-_i(U)} \mu_{ij}x_{ij} = \begin{cases} x_i sign(i), & i \in I^* \\ a_i, & i \in I\setminus I^* \end{cases}$$

$$(3) \qquad \sum_{(i,j)\in U} \lambda_{ij}^p x_{ij} + \sum_{i\in I^*} \lambda_i^p x_i = \beta^p \, , \, p = \overline{1,q}$$

$$(4) \qquad 0 \le x_{ij} \le d_{ij}, \quad (i,j) \in U, \quad b_{*i} \le x_i \le b_i^*, \quad i \in I^*$$

Vector $x = (x_{ij}, (i,j) \in U, x_{i,}, i \in I^*)$ is called a plan, if it satisfies restrictions (2) - (4), ($x \in X$ – a set of plans). The plan $x^0 \in X$ is optimal, if $c'x^0 = \min c'x, x \in X, c = (c_{ij}, (i,j) \in U, c_i, i \in I^*)$. A suboptimal plan $x^\varepsilon = (x_{ij}^\varepsilon, (i,j) \in U, x_i^\varepsilon, i \in I^*$ is defined by the inequality $c'x^\varepsilon - c'x^0 \le \varepsilon$, $x^\varepsilon \in X$, where $\varepsilon \ge 0$ is a given precision. The totality $\{x, K\}$ of a plan x and a support K is called a basic plan. The pair $K = \{U_K, I_K^*\}$ can be represented as a direct sum of $R = \{U_R, I_R^*\}$ and $W = \{U_W, I_W^*\}$ with the properties given in [1, 2, 5]. Let $N = \{U_N, I_N^*\}$, $U_N = U \setminus U_K$, $I_N^* = I^* \setminus I_K$.

2. The dual problem.

The dual problem for problem (1) - (4) looks like

$$\sum_{i\in I\setminus I^*} a_i y_k + \sum_{p=1}^{q} \beta^p \tau_p + \sum_{i\in I^*} a_{*i}\omega_i - \sum_{i\in I^*} a_i^* t_i - \sum_{(i,j)\in U} d_{ij} v_{ij} \longrightarrow \max,$$

$$(5) \qquad \begin{gathered} y_i - \mu_{ij} y_j - v_{ij} + \sum_{p=1}^{q} \lambda_{ij}^p \tau_p \le c_{ij}, \, (i,j) \in U, \\ -y_i sign(i) + \omega_i - t_i + \sum_{p=1}^{q} \lambda_{ij}^p \tau_p = c_i, \, i \in I^*, \\ v_{ij} \ge 0, \, (i,j) \in U, \quad t_i \ge 0, \quad \omega_i \ge 0, \quad i \in I^*. \end{gathered}$$

Vector $\lambda = (y, \tau, t, \omega, v)$, $y = (y_i, i \in I)$, $\tau = (\tau_p, p = \overline{1,q})$, $t = (t_i, i \in I^*)$, $\omega = (\omega_i, i \in I^*)$, $\nu = (v_{ij}(i,j) \in U)$, satisfying all the restrictions of problem (5), is called a dual plan, $\lambda \in \Lambda$, where Λ is the set of dual plans. For every dual plan $\lambda \in \Lambda$ we consider a corresponding co-plan δ, $\delta = (\delta_{ij}, (i,j) \in U, \delta_i, i \in I^*)$:

$$\delta_{ij} = c_{ij} - y_i + \mu_{ij} y_j - \sum_{p=1}^{?} \lambda_{ij}^p r_p, \, (i,j) \in U, \, \delta_i = c_i + y_i sign(i) - \sum_{p=1}^{q} \lambda_i^p r_p, \, i \in I^*$$

In the support methods, that use the adaptive norm [4], accompanying co-plans are defined as follows:

$$\delta_{ij} = \widetilde{\Delta}_{ij},\ (i,j) \in U, \quad \delta_i = \widetilde{\Delta}_i, \quad i \in I^*,$$

$$\widetilde{\Delta}_{\tau\rho} = \Delta_{\tau\rho} - \sum_{p=1}^{q} r_p \Lambda^p_{\tau\rho}, \quad (\tau,\rho) \in U_N, \quad \widetilde{\Delta}_\gamma = \Delta_\gamma - \sum_{p=1}^{q} r_p \Lambda^p_\gamma, \quad \gamma \in I^*_N,$$

$$\Delta_{\tau\rho} = c_{\tau\rho} + \sum_{(i,j)\in U_R} c_{ij}\delta^{\tau\rho}_{ij} + \sum_{i\in U^*_R} c_i \delta^{\tau\rho}_i, \quad \Delta_\gamma = c_\gamma + \sum_{(i,j)\in U_R} c_{ij}\delta^{\gamma}_{ij} + \sum_{i\in U^*_R} c_i \delta^{\gamma}_i$$

$$r' = \Delta'_W \Lambda^{-1}_W,\ r = (r_1, r_2, \ldots, r_q),\ r = \tau = (\tau_p,\ p = \overline{1,q}),\ \Lambda_W = (\Lambda_{W_1}, \Lambda_{W_2}),$$

$$\Lambda_{W_1} = (\Lambda^p_{\tau\rho},\ p = \overline{1,q};\ (\tau,\rho) \in U_W), \quad \Lambda_{W_2} = (\Lambda^p_\gamma,\ p = \overline{1,q},\ \gamma \in I^*_W),$$

$$\Lambda^p_{\tau\rho} = \sum_{(i,j)\in U_R} \lambda^p_{ij}\delta^{\tau\rho}_{ij} + \sum_{i\in U^*_R} \lambda^p_i \delta^{\tau\rho}_i + \lambda^p_{\tau\rho}, \quad \Lambda^p_\gamma = \sum_{(i,j)\in U_R} \lambda^p_{ij}\delta^{\gamma}_{ij} + \sum_{i\in U^*_R} \lambda^p_i \delta^{\gamma}_i + \lambda^p_\gamma$$

The columns of matrices $S_W = (\delta_{\tau\rho}, (\tau,\rho) \in U_W;\ \delta_\gamma,\ \gamma \in I^*_W)$ and $S_N = (\delta_{\tau\rho}, (\tau,\rho) \in U_N;\ \delta_\gamma,\ \gamma \in I^*_N)$, being put together, form the basis of the solution space for the homogeneous system corresponding to (2) [1]. Since K is a support, $\det \Lambda_W \neq 0$ [1].

Further we consider only accompanying co-plans. Alongside with the co-plan δ, we introduce another co-plan $\bar{\delta} = (\delta_{ij} + \triangle\delta_{ij},\ (i,j) \in U;\ \delta_i + \triangle\delta_i,\ i \in I^*)$,

$$\triangle\delta_{ij} = -(\triangle\delta_i - \mu_{ij}\triangle\delta_j - \sum_{p=1}^{q} \lambda^p_{ij}\triangle\tau_p,\ (i,j) \in U),$$

$$\triangle\delta_i = \triangle y_i sign(i) - \sum_{p=1}^{q} \lambda^p_i \triangle\tau_p,\ i \in I.$$

3. Decomposition of linear system for pseudo-plan calculation.

Let $\{\delta, K\}$ be a support co-plan. We construct the pseudo-plan $æ = (æ_{ij}, (i,j) \in U;\ æ_i,\ i \in I^*)$, applying the algorithms for linear systems decomposition [1] and using the support co-plan $\{\delta, K\}$ as follows. We construct the components of the vector $æ_N = (æ_{ij}, (i,j) \in U_N;\ æ_i,\ i \in I^*_N)$ satisfying the optimality criterion [1, 2] :

$æ_{ij} = 0$, if $\widetilde{\Delta}_{ij} > 0$; $\quad æ_i = b_{*i}$, if $\widetilde{\Delta}_i > 0$;

$æ_{ij} = d_{ij}$, if $\widetilde{\Delta}_{ij} < 0$; $\quad æ_i = b^*_i$, if $\widetilde{\Delta}_i < 0$;

$æ_{ij} \in [0, d_{ij}]$, if $\widetilde{\Delta}_{ij} = 0$; $(i,j) \in U_N$; $\quad æ_i \in [b_{*i}, b^*_i]$, if $\widetilde{\Delta}_i = 0$; $(i,j) \in U_N$.

Since K is a support, $\det \Lambda_W \neq 0$. The components of $æ_W = (æ_{\tau\rho}, (\tau,\rho) \in U_W;\, æ_\gamma,\, \gamma \in I^*_W)$ are uniquely computed from the system:

$$\Lambda_W æ_W = \widetilde{\beta},$$

$$\widetilde{\beta} = (\widetilde{\beta}^p, p = \overline{1,q}),\ \widetilde{\beta}^p = \widetilde{\alpha}^p - \sum_{(\tau,\rho)\in U^*_N} \Lambda^p_{\tau\rho} æ_{\tau\rho} - \sum_{\gamma\in U_N} \Lambda^p_\gamma æ_\gamma,$$

$$\widetilde{\alpha}^p = \beta^p - \sum_{(i,j)\in U_R} \lambda^p_{ij} \widetilde{æ}_{ij} - \sum_{k\in I^*_R} \lambda^p_k \widetilde{æ}_k,\ p = \overline{1,q},$$

where $\widetilde{æ} = (\widetilde{æ}_{ij}, (i,j) \in U;\, \widetilde{æ}_i,\, i \in I^*)$ – any particular solution of system (2). Using network properties of the solution space basis [1, 3, 5] of the homogeneous system corresponding to (2), we compute the components of vector $æ_R = (æ_{\tau\rho}, (\tau,\rho) \in U_R;\, æ_\gamma,\, \gamma \in I^*_R)$:

$$æ_{ij} = \sum_{(\tau\rho)\in U\setminus U_R} æ_{\tau\rho}\delta^{\tau\rho}_{ij} + \sum_{\gamma\in I^*\setminus I^*_R} æ_\gamma \delta^\gamma_{ij} + \widetilde{æ}_{ij},\ (i,j)\in U_R,$$

$$æ_i = \sum_{(\tau\rho)\in U\setminus U_R} æ_{\tau\rho}\delta^{\tau\rho}_i + \sum_{\gamma\in I^*\setminus I^*_R} æ_\gamma \delta^\gamma_i + \widetilde{æ}_i, i \in I^*_R$$

The worst case complexity of finding a partial solution is $O(|I| + |I^*|)$.

4. Numerical experiments.

The key parameters of the problems are submitted in Table 1. Results of numerical experiments of the solution of linear systems with the use of the stated principles of decomposition are submitted in Table 2. At construction of the particular solution by means of the package " MATLAB 6.1" for achievement of the maximal productivity the built - in functions of a nucleus and technology of rarefied matrixes were used. Testing was carried out on a personal computer of the following configuration: the CENTRAL PROCESSING UNIT - Duron 1000 MHz, the RAM - 256 Mb, OS - Windows 2000. Time of construction of solutions is specified in seconds. The symbol " - " means, for construction of the system solution of specified size with the use of the package " MATLAB 6.1" not enough operative memory (Out Of Memory error).

5. Linear system decomposition algorithms for dual suitable direction calculation.

If the support K is not an optimum [1, 2] and the support plan $\{x, K\}$ is dually nonsingular ($\delta_{ij} \neq 0$, $(i,j) \in U_N$, $\delta_i \neq 0$, $i \in I^*_N$), then there exists a

Table 1: Table 1. The input data.

Number of the Problem	Number of Nodes	Number of Arcs	Number of Variable Intensities	Number of Unknowns	Number of Additional Restrictions
1	100	396	10	3974	20
2	300	2431	20	2451	128
3	500	2903	0	2903	50
4	1500	4859	0	4859	100
5	1000	50912	0	50912	10
6	8000	72477	0	72477	10
7	10000	119781	9	119790	64
8	30000	262848	0	262848	10
9	2000	402206	0	402206	10
10	10000	510559	0	510559	10

Table 2: Table 2. Results of numerical experiments.

Number of the Problem	Time for Construction of the Support (C++)	Time for Construction of the Particular Solution (C++)	Time for Construction of the Particular Solution (MATLAB)
1	0	0,01	4,12
2	0	0,17	5,29
3	0	0,05	5,68
4	0,01	0,31	70,24
5	0,03	0,211	-
6	0,06	1,33	-
7	0,081	22,84	-
8	0,2	5,29	-
9	0,31	5,07	-
10	0,4	7,45	-

variation $\triangle\delta$ of the co-plan δ which leads to an increase in the dual objective function [4]. As a result of changing one support component $\delta_{i_0j_0}$ or δ_{i_0}, the nonsupport components of the increment $\triangle\delta_N = (\triangle\delta_{ij},\, (i,j) \in U_N,\, \triangle\delta_i,\, i \in I_N^*)$ change, too. In the adaptive algorithms [4] two variants for construction of a dual suitable direction $\triangle\delta$ for changing the dual objective function are possible.

1) Let (i_0, j_0) be a critical arc (on which the limit of the flow $\overline{x}_{ij}$ $(0 \vee d_{i_0j_0})$ is achieved). We construct the support components of the direction $\triangle\delta_K = (\triangle\delta_{ij},\, (i,j) \in U_K;\, \triangle\delta_i,\, i \in I_K^*$ as follows:

$$\triangle\delta_{i_0j_0} = \begin{cases} -1, \text{ if } \ \overline{x}_{i_0j_0} = d_{i_0j_0}, \\ 1, \text{ if } \ \overline{x}_{i_0j_0} = 0, \end{cases} \triangle\delta_{ij} = 0,\ (i,j) \in U_K \setminus (i_0, j_0),\ \triangle\delta_i = 0,\ i \in I_K^*$$

2) Let i_0 be a critical node (on which the limit of the component is achieved $\overline{x}_{i_0}$ $(\overline{x}_{i_0} = b_{*_i} \vee b_i^*)$). We construct the support components of the direction $\Delta\delta_K = (\Delta\delta_{i_j}, (i,j) \in U_K;\, \Delta\delta_i, i \in I_K^*)$ by the rules:

$$\triangle\delta_{i_0} = \begin{cases} -1, \text{ if } \ \overline{x}_{i_0} = b_{i_0}^*, \\ 1, \text{ if } \ \overline{x}_{i_0} = b_{*i_0}, \end{cases} \quad \triangle\delta_{ij} = 0,\ (i,j) \in U_K,\ \triangle\delta_i = 0,\ i \in I_K^* \setminus i_0$$

We put $\alpha = \Delta\delta_{i_0j_0}$ if arc (i_0, j_0) is critical. If node i_0 is critical, we put $\alpha = \Delta\delta_{i_0}$. We consider case 1). The support components $\Delta\delta_K = (\Delta\delta_{i_j}, (i,j) \in U_K;\, \Delta\delta_i, i \in I_K^*)$ of the increment $\Delta\delta = (\Delta\delta_{i_j}, (i,j) \in U_K;\, \Delta\delta_i, i \in I_K^*)$ satisfy the system:

$$\triangle\delta_{i_0j_0} = -(\triangle u_{i_0} - \mu_{i_0j_0} \triangle u_{j_0} + \sum_{p=1}^{q} \lambda_{ij}^p \Delta r_p) = \alpha,$$

$$\text{(6)} \qquad \triangle\delta_{ij} = -(\triangle u_i - \mu_{ij} \triangle u_j + \sum_{p=1}^{q} \lambda_{ij}^p \Delta r_p) = 0,\ (i,\, j) \in U_K \setminus (i_0, j_0),$$

$$\triangle\delta_i = \triangle u_i sign(i) - \sum_{p=1}^{q} \lambda_i^p \Delta r_p) = 0,\ \ i \in I_K^*$$

In the case of 2), support components
$\Delta\delta_K = (\Delta\delta_{i_j}, (i,j) \in U_K;\, \Delta\delta_i, i \in I_K^*)$ of the increment

$\Delta\delta = (\Delta\delta_{i_j}, (i,j) \in U_K;\ \Delta\delta_i, i \in I_K^*)$ satisfy the system:

$$\triangle\delta_{i_0} = \triangle u_{i_0} sign(i_0) - \sum_{p=1}^{q} \lambda_{i_0}^p \Delta r_p = \alpha$$

(7) $$\triangle\delta_{ij} = -(\triangle u_i - \mu_{ij} \triangle u_j + \sum_{p=1}^{q} \lambda_{ij}^p \Delta r_p) = 0,\ (i,\ j) \in U_K,$$

$$\triangle\delta_i = \triangle u_i sign(i) - \sum_{p=1}^{q} \lambda_i^p \Delta r_p = 0,\ i \in I_K^* \backslash i_0$$

Let us introduce effective algorithms for constructing the solution of systems (6), (7). These algorithms are based on the principles of decomposition of system equations in its network and general parts [5]. We consider elements from the totality of sets $W = (U_W, I_W^*)$. To every element $(\tau,\rho) \in U_W$ or $\gamma \in I_W^*$ there corresponds a characteristic vector $\delta(\tau,\rho)$ or $\delta(\gamma)$, $\delta(\tau,\rho) = (\delta_{ij}^{\tau\rho}, (i,j) \in U;$ $\delta_i^{\tau\rho}, i \in I^*)$, $\delta(\gamma) = (\delta_{ij}^{\gamma}, (i,j) \in U; \delta_i^{\gamma}, i \in I^*)$ [5]. We perform the following transformations of equations (6). Consider a characteristic vector $\delta(\tau,\rho)$. We multiply every equation of system (6), corresponding to arc $(i,j) \in U_K$, by a nonzero component $\delta_{ij}^{\tau\rho} \neq 0$ of vector $\delta(\tau,\rho)$. Similarly, every equation of system (6) which corresponds to node $i \in I_K^*$ is multiplied by a nonzero component $\delta_i^{\tau\rho} \neq 0$ of the vector $\delta(\tau,\rho)$. Then we add up the received equalities. For any characteristic vector $\delta(\gamma) = (\delta_{ij}^{\gamma}, (i,j) \in U; \delta_i^{\gamma}, i \in I^*)$, generated by node γ, we multiply every equation of system (6) which corresponds to an arc $(i,j) \in U_K$, by a nonzero component $\delta_{ij}^{\gamma} \neq 0$ and, every equation of system (6) which corresponds to a node $i \in I_K^*$, we multiply by a nonzero component $\delta_i^{\gamma} \neq 0$ of vector $\delta(\gamma)$. Then we sum up the received equalities. Let us perform the specified transformations of system (6) for all elements of the totality of sets $W = (U_W, I_W^*)$:

(8) $$\sum_{p=1}^{q} \Lambda_{\tau\rho}^p \Delta r_p = -\alpha\delta_{i_0j_0}^{\tau\rho}, (\tau,\rho) \in U_W, \sum_{p=1}^{q} \Lambda_{\gamma}^p \Delta r_p = -\alpha\delta_{i_0j_0}^{\gamma}, \gamma \in I_W^*,$$

$$\Lambda_{\tau\rho}^p = \sum_{(i,j)\in U_R} \lambda_{ij}^p \delta_{ij}^{\tau p} + \sum_{i\in I_R^*} \lambda_i^p \delta_i^{\tau p} + \lambda_{\tau\rho}^p, \Lambda_{\gamma}^p = \sum_{(i,j)\in U_R} \lambda_{ij}^p \delta_{ij}^{\gamma} + \sum_{i\in I_R^*} \lambda_i^p \delta_i^{\gamma} + \lambda_{\gamma}^p.$$

Let us present (8) in a matrix-vector form:

(9) $$\Lambda_W' \Delta r = \overline{\alpha}, \quad \overline{\alpha} = (-\alpha\delta_{i_0j_0}^{\tau\rho}, (\tau,\rho) \in U_W; -\alpha\delta_{i_0j_0}^{\gamma}, \gamma \in I_W^*),$$

$$\Lambda_W = (\Lambda_{W_1}, \Lambda_{W_2}), \quad \Lambda_{W_1} = (\Lambda^{\rho}_{\tau\rho}, \rho = \overline{1,q}; (\tau,\rho) \in U_W),$$
$$\Lambda_{W_2} = (\Lambda^{\rho}_{\gamma}, \rho = \overline{1,q}, \gamma \in I^*_W),$$

The totality of sets $K = \{U_K, I^*_K\}$ is a support of network G for system (2), (3), hence, matrix Λ_W is nonsingular. From system (9) we compute Δr, $\Delta r = (\Lambda'_W)^{-1}\overline{\alpha}$.

Consider system (7). We perform transformations of system (7) for all elements of the totality $W = (U_W, I^*_W)$:

$$\text{(10)} \qquad \sum_{p=1}^{q} \Lambda^p_{\tau\rho}\Delta r_p = -\alpha\delta^{\tau\rho}_{i_0}, (\tau,\rho) \in U_W, \sum_{p=1}^{q} \Lambda^p_{\gamma}\Delta r_p = -\alpha\delta^{\gamma}_{i_0}, \gamma \in I^*_W,$$

Let us represent (10) in a matrix-vector form:

$$\text{(11)} \qquad \Lambda'_W \Delta r = \overline{\alpha}, \quad \overline{\alpha} = (-\alpha\delta^{\tau\rho}_{i_0}, (\tau,\rho) \in U_W; -\alpha\delta^{\gamma}_{i_0}, \gamma \in I^*_W)$$

The totality of sets $K = \{U_K, I^*_K\}$ is a support of the network G for system (2), (3), and hence, matrix Λ_W is nonsingular. From system (11) we compute Δr, $\Delta r = (\Lambda'_W)^{-1}\overline{\alpha}$.

Components of vector $\Delta\delta_N = (\Delta\delta_{\tau\rho}, (\tau,\rho) \in U_N; \Delta\delta_\gamma, \gamma \in I^*_N)$ are computed from the relation $\Delta\delta_N = -\Lambda'_N\Delta r - \alpha t$, where $t = (\delta^{\tau\rho}_{i_0 j_0}, (\tau,\rho) \in U_N; \delta^{\gamma}_{i_0 j_0}, \gamma \in I^*_N)$, if arc (i_0, j_0) is critical. If node i_0 is critical, vector $t = (\delta^{\tau\rho}_{i_0}, (\tau,\rho) \in U_N; \delta^{\gamma}_{i_o}, \gamma \in I^*_N)$, $\Lambda_N = (\Lambda_{N_1}, \Lambda_{N_2})$, $\Lambda_{N_1} = (\Lambda^{\rho}_{\tau\rho}, \rho = \overline{1,q}; (\tau,\rho) \in U_N)$, $\Lambda_{N_2} = (\Lambda^{\rho}_{\gamma}, \rho = \overline{1,q}, \gamma \in I^*_N)$.

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