Computing 76, 353–357 (2006)
Digital Object Identifier (DOI) 10.1007/s00607-005-0138-y

Computing
Printed in Austria

Fractal-like Matrices

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Received February 15, 2005; revised June 16, 2005
Published online: November 3, 2005

Abstract

We introduce special sparse $n \times n$-matrices ($n = 2^k$) containing up to $3^k \approx n^{1.585}$ nonzero elements. Their structure is inherited from the famous Sierpinski triangle and is not sensitive to matrix multiplication and inversion. The arithmetical complexity of taking product or inverse of such matrices is proved to be $O(n^2)$.

AMS Subject Classifications: 16S50, 28A80, 65F50.

Keywords: Matrix multiplication and inversion, sparse matrices, matrix rings, sierpinski triangle.

1. Introduction

The currently best lower bound for matrix multiplication algorithm complexity is $\Omega(n^2)$. Still there is little progress towards reaching (or rising) it [1], [2]. The ordinary algorithm with $\Theta(n^3)$ complexity is most commonly used, because asymptotically faster algorithms are impractical for small n. Taking matrix inverse is also hard: in terms of computational complexity it is equivalent to matrix multiplication [3].

For instance, let us consider the set of band matrices. They have zeros outside a diagonal stripe of fixed width (see Fig. 1):

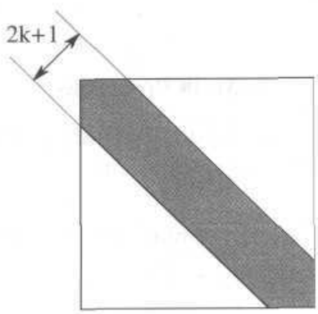

Fig. 1. Structure of band matrices

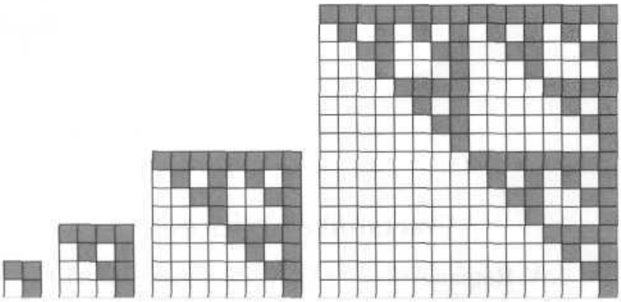

Fig. 2. Structure of fractal-like matrices

Strictly speaking, $A = (a_{ij}) \in \mathbb{R}_{n\times n}$ is a band matrix iff $\exists k$ such that $a_{ij} = 0$ if $|i - j| > k$. Typically one assumes $1 < k \ll n$. Band matrices allow multiplication and inversion in $O(k^2 n)$ time. Indeed, we should use the ordinary multiplication algorithm or Gauss method correspondingly, taking into account specific zero layout. However, if $k > 1$ almost every matrix from this set doubles its band width after one performs multiplication or inversion on it. If $k = O(\sqrt{n})$ we are able to take products and inverses, using $O(n^2)$ arithmetical operations.

In the current work we build another set of special sparse matrices. Unlike band matrices they have a structure that is invariant under multiplication and inversion. Also they may hold asymptotically more than $n\sqrt{n}$ nonzero entries and still be multiplied or inverted in $O(n^2)$ time.

2. Definition of Fractal-like Matrices

We define F_n (the set of n-ordered fractal-like matrices) for each n that is a power of 2, using mathematical induction:

(1) $F_{2^0} := \mathbb{R}_{1\times 1}$;

(2) $\forall k > 0,\ F_{2^k} := \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} : A, B, C \in F_{2^{k-1}} \right\}$.

If $n \to \infty$ the structure of F_n elements reminds Sierpinski triangle, the famous fractal [4] (see Fig. 2).

3. Main Properties

The set F_n is a linear subspace and a subring of $\mathbb{R}_{n\times n}$ therefore it is a subalgebra of $\mathbb{R}_{n\times n}$. Indeed, let's prove the following theorem.

Theorem: F_n is an algebra over $\mathbb{R}$, containing E_n. The nonsingular entries of F_n are invertible and their inverses belong to F_n, too. $\dim F_n = n^{\log_2 3}$.

Proof: Let us use mathematical induction.

Table 1. Arithmetical cost of operations over fractal-like matrices. ADD/SUB – real addition/subtraction, MUL – real multiplication, DIV – real division, NEG – real sign change

Matrix operation	ADD/SUB count	MUL count	DIV count	NEG count
Addition/subtraction	$n^{\log_2 3}$	0	0	0
Multiplication	$n^2 - n^{\log_2 3}$	n^2	0	0
Inversion	$n^2 - 2n^{\log_2 3} + n$	$n^2 - n$	n	$n^{\log_2 3} - n$

(1) Since $\mathbb{R}_{1\times 1} \cong \mathbb{R}$ the theorem statement is true for $n = 1$.
(2) Let us prove that it is also true for $n = 2^k, k > 0$, assuming that we have already made a proof for $n = 2^{k-1}$.

Consider two matrices $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, B = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix} \in F_n$.

$$\forall \alpha, \beta \in \mathbb{R}, \alpha A + \beta B = \begin{pmatrix} \alpha A_1 + \beta B_1 & \alpha A_2 + \beta B_2 \\ 0 & \alpha A_3 + \beta B_3 \end{pmatrix} \in F_n,$$

because $\alpha A_i + \beta B_i \in F_{\frac{n}{2}}, i = 1, 2, 3.$

$$AB = \begin{pmatrix} A_1 B_1 & A_1 B_2 + A_2 B_3 \\ 0 & A_3 B_3 \end{pmatrix} \in F_n, \text{ because}$$

$A_1 B_1, A_1 B_2 + A_2 B_3, A_3 B_3 \in F_{\frac{n}{2}}$.

$$E_n = \begin{pmatrix} E_{\frac{n}{2}} & 0 \\ 0 & E_{\frac{n}{2}} \end{pmatrix} \in F_n, \text{ because } 0_{\frac{n}{2}}, E_{\frac{n}{2}} \in F_{\frac{n}{2}}.$$

Finally, if $\det A = \det A_1 \cdot \det A_3 \neq 0$ then

$$A^{-1} = \begin{pmatrix} A_1^{-1} & -A_1^{-1} A_2 A_3^{-1} \\ 0 & A_3^{-1} \end{pmatrix} \in F_n,$$

because $A_1^{-1}, A_3^{-1}, -A_1^{-1} A_2 A_3^{-1} \in F_{\frac{n}{2}}$.

$$\dim F_n = 3(\frac{n}{2})^{\log_2 3} = n^{\log_2 3}.$$

□

To obtain the number of arithmetical operations required to add, multiply or invert fractal-like matrices through recursive application of above formulas, one may solve recurrence equations. For example, the equation $M_k = 4M_{k-1}$ (for the number of real multiplications required to take product in F_{2^k}) under the initial condition $M_0 = 1$ has the following solution: $M_k = 4^k = n^2$. More accurate analysis leads to the results listed below.

Let us show that in F_n multiplication is as hard as inversion (the same happens in $\mathbb{R}_{n\times n}$ [3]). Indeed, multiplication of A by B may be reduced to 2 inversions:

$$\begin{pmatrix} E & A \\ 0 & -B^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} E & AB \\ 0 & -B \end{pmatrix}$$

(if B is singular we may assume $B := B + \lambda E$ to make it diagonally dominant). As for inversion, it is reduced to 2 multiplications, 2 inversions and 1 sign change of lower ordered matrices (see the formula in the above proof). Suppose there exists a multiplication algorithm with $O(n^\alpha)$ complexity ($\alpha \geq \log_2 3$). Then we can apply it recursively and finally invert matrices, using $O(n^\alpha)$ operations. It follows from the corresponding recurrence equations for operation count.

Now let us consider the following equality for product of a fractal-like matrix by a vector:

$$Ab = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} A_1 b_1 + A_2 b_2 \\ A_3 b_2 \end{pmatrix}.$$

The calculation under this formula demands $n^{\log_2 3}$ real multiplications and $n^{\log_2 3} - n$ real additions.

As for solving a linear equation $Ax = b$, $A \in F_n$ the formula

$$x = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} A_1^{-1}(b_1 - A_2(A_3^{-1} b_2)) \\ A_3^{-1} b_2 \end{pmatrix}$$

yields an algorithm with n divisions, $n^{\log_2 3} - n$ multiplications and $n^{\log_2 3} - n$ additions/subtractions.

4. Optimal Storage

We propose a special storage scheme for fractal-like matrices that preserves memory wasting and is convenient to perform arithmetical operations on them. Instead of remembering a two-dimensional array of size n^2 we may store a row of length $n^{\log_2 3}$. Indeed,

(1) $(a) \in F_1$ should be represented by a vector $\mathbf{a} = [a]^T$ of length $1 = 1^{\log_2 3}$;

(2) $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in F_n$, $n = 2^k$, $k > 0$ should be represented by the vector $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^T$ of length $3(\frac{n}{2})^{\log_2 3} = n^{\log_2 3}$, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are representations of matrices A, B and C correspondingly.

Example: The matrix $\begin{pmatrix} 1 & 2 & 4 & 5 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 9 \end{pmatrix}$ should be represented by the vector $[1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9]^T$.

Acknowledgements

We thank Dr. V. I. Repnikov for useful conversations and interesting suggestions.

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